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The Generalization of Mathematical Morphology to Non-numeric Sets

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Abstract. It is well known that mathematical morphology plays an important role in image analysis as it enables locating and detecting shapes as well as noise filtering. This paper shows how many of the important properties in mathematical morphology hold in a much more general setting of symbolic or non-numeric sets. This includes the operations of dilation, erosion, opening and closing. For example, dilation of a union is the unions of dilations. Dilation is a union preserving operation. Erosion of an intersection is an intersection of erosions. Erosion is an intersection preserving operation. If A is a subset of B, then the dilation of A is a subset of the dilation of B and the erosion of A is a subset of the erosion of B. There is a duality between dilation and erosion. Openings are formed by an erosion followed by dilation. Closings are formed by a dilation followed by erosion. Openings are idempotent: doing it more than once is the same as doing it once. Closings also are idempotent. And there are other properties of mathematical morphology that hold in the setting of arbitrary sets. Further that properties like idempotence of openings and closings happen in a setting of general sets whose elements are not numerical and where there are no numerical calculations and no orderings is surprising and unexpected.

keywords: dilation, erosion, set operator, increasing operator, decreasing operator, expansive operator, contractive operator, union preserving operator, intersection preserving operator, set dilation operator ,set erosion operator,dual operator, adjoint operator, opening operator, closing operator.

1 Introduction

Mathematical morphology plays an important role in image analysis as it enables locating and detecting shapes as well as noise filtering. This paper shows how many of the important properties in mathematical morphology hold in a much more general setting of symbolic or non-numeric sets. This includes the operations of dilation, erosion, opening and closing. For example, dilation of a union is the unions of dilations. Dilation is a union preserving operation. Erosion of an intersection is an intersection of erosions. Erosion is an intersection preserving operation. If A is a subset of B, then the dilation of A is a subset of the dilation of B and the erosion of A is a subset of the erosion of B. There is a duality between dilation and erosion. Openings are formed by an erosion followed by dilation. Closings are formed by a dilation followed by erosion. Openings and closings are idempotent: doing it more than once is the same as doing it once.

In section 2, we describe the basic definitions and properties of the different set operators including increasing operators, decreasing operators, expansive operators, contractive operators, union preserving operators, intersection preserving operators, set dilation operators, set erosion operators, dual operators, opening operators and closing operators.

In section 3, we describe the basic definitions and properties of the inverses of set operators, including the inverses of the union preserving operators, the inverses of intersection preserving operators, the inverses of set dilation operators, and the inverses of set erosion operators. A complete lattice of all the subsets of a set gives rise to the possibility of an inverse operator. And it is impossible to have it in the general case of the lattice.

In section 4, we define 10 theorems to construct the closing and opening operators by using the set operators and the inverses we defined in the last two sections.

The state of the art of Mathematical Morphology theory is Complete Lattice operators. Such framework was introduced by Serra and Matheron ($[1,2]$) in the eighties, with many contributions for the class of increasing operators. Since then, many researchers have contributed extensions of this theory. In the 1990's, Banon and Barrera developed general lattice operator representations, general set mapping representations, in particular they used the concepts of kernel and basis to prove that any set mapping (not necessarily translation invariant) can be decomposed by a set of non-translation invariant sup-generating operators, or, dual, non-translation invariant inf-generating operators. See Banon and Barrera for a more complete review of the earlier work ($[3,4]$).

All the previous work is more general because it's on a general lattice. The work we did is a specialization of the work that has already been published.

Our work is easier to understand since we define our definitions on power sets. We do not involve the concepts of completed lattices, translation invariant mapping, kernel mapping, basis mapping etc. All of what has we do is restricted to finite sets. Because everything in our domain is countable, we do not get into the infimum and supremum. Therefore, our definition and proofs are easier to understand, and the proofs are more direct. In addition, our notation is simpler. We have developed many new properties and theorems.

2 Basic Definitions and Properties

We do not review papers of mathematical morphology or its extensions to lattices. See([5, 6, 8–27]). There is not enough space to do that and describe our exciting generalization of mathematical morphology. There is so much to say. We begin with basics.

Definition 1. *A universal set is a finite set that contains arbitrary non-numeric elements. We designate whatever universal set we are working with by U. Subsets of U will be denoted by capital letters. Individual elements of U will be denoted by lower case letters.[24]*

For example, we may have $U = \{a, b, c, d, ...p, q\}$

Definition 2. *The power set of U is the collection of all subsets of U, including the empty set and is denoted by $\mathcal{P}(U)$.*

For example given a universal set $U = \{a, b, c\}$, there are eight possible subsets: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$ and this constitutes $\mathcal{P}(U)$.

Definition 3. *A set operator $\mathcal{F}$ is a function $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$. We follow the convention that set operators will be denoted by calligraphic upper case letters such as $\mathcal{A}$ and $\mathcal{B}$.*

Definition 4. *The operator composition of a set operator $\mathcal{F}$ with another set operator $\mathcal{G}$ where $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ will be denoted by $\mathcal{G} \odot \mathcal{F}$ and it means first apply $\mathcal{F}$ and then apply $\mathcal{G}$. As appropriate, if we are composing one set operator with another acting on a set A, we may write it as $\mathcal{G}(\mathcal{F}(A))$.*

Proposition 1. *Set operator composition is associative.*

$$\mathcal{F} \odot (\mathcal{G} \odot \mathcal{H}) = (\mathcal{F} \odot \mathcal{G}) \odot \mathcal{H}$$

2.1 Increasing and Decreasing Operators

Definition 5. *A set operator $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ is said to be increasing if and only if $A \subseteq B$ implies $\mathcal{F}(A) \subseteq \mathcal{F}(B)$. The operator $\mathcal{F}$ is said to be decreasing if and only if $A \subseteq B$ implies $\mathcal{F}(A) \supseteq \mathcal{F}(B)$.[7, 19]*

Example 1. *Given a universal set domain $U = \{a, b, c\}$. Let $\mathcal{A} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an increasing operator, and $\mathcal{B} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a decreasing operator with following mappings as shown in Table 1.*

Table 1. Illustrates an increasing operator $\mathcal{A}$ and a decreasing operator $\mathcal{B}$ defined on the power set of $\{a, b, c\}$

S	$\mathcal{A}(S)$	S	$\mathcal{B}(S)$
$\emptyset$	$\{a\}$	$\emptyset$	$\{a,b,c\}$
$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$
$\{b\}$	$\{a,c\}$	$\{b\}$	$\{b\}$
$\{c\}$	$\{a\}$	$\{c\}$	$\{a,b,c\}$
$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{b\}$
$\{a,c\}$	$\{a,b\}$	$\{a,c\}$	$\{a\}$
$\{b,c\}$	$\{a,c\}$	$\{b,c\}$	$\{b\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\emptyset$

Proposition 2. *(1) If $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ are increasing operators, then $\mathcal{H} = \mathcal{F} \odot \mathcal{G}$ is an increasing operator.*

(2) If $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ are decreasing operators, then $\mathcal{H} = \mathcal{F} \odot \mathcal{G}$ is an increasing operator.

(3) If $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ is a decreasing operator and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ is an increasing operator, then $\mathcal{H} = \mathcal{F} \odot \mathcal{G}$ is an decreasing operator.

(4) If $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ is an increasing operator and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ is a decreasing operator, then $\mathcal{H} = \mathcal{F} \odot \mathcal{G}$ is a decreasing operator.

Proposition 3. *Let $\mathcal{F}$ be an increasing operator on U and let $A, B \subseteq U$. Then*

1. $\mathcal{F}(A \cup B) \supseteq \mathcal{F}(A) \cup \mathcal{F}(B)$
2. $\mathcal{F}(A \cap B) \subseteq \mathcal{F}(A) \cap \mathcal{F}(B)$

2.2 Expansive and Contractive Operators

A set operator is able to take a set and produce a related set that includes the original set. That kind of set operator is called an expansive operator. The set operator that takes a set and produces a related set that excludes some of the original set is called a contractive operator. Simply, we say that the expansive operators produce sets which are super sets, the contractive operators produce sets which are subsets.

Definition 6. *An operator $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ is said to be an expansive if and only if $A \subseteq \mathcal{F}(A)$. The operator $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ is said to be contractive if and only if $\mathcal{F}(A) \subseteq A$.[19]*

Example 2. *As shown in Table 2, given a universal set domain $U = \{a, b, c\}$. $\mathcal{C} : \mathcal{P}(U) \to \mathcal{P}(U)$ is an expansive operator and $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ is a contractive operator with the following mappings.*

Table 2. Illustrates an expansive operator $\mathcal{C}$ and a contractive operator $\mathcal{D}$ defined on the power set of $\{a, b, c\}$

S	$\mathcal{C}(S)$	S	$\mathcal{D}(S)$
$\emptyset$	$\{b\}$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a, b\}$	$\{a\}$	$\emptyset$
$\{b\}$	$\{a, b\}$	$\{b\}$	$\emptyset$
$\{c\}$	$\{a, c\}$	$\{c\}$	$\{c\}$
$\{a, b\}$	$\{a, b, c\}$	$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{a, b, c\}$	$\{a, c\}$	$\{c\}$
$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{b, c\}$

Proposition 4. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$. If $\mathcal{F}$ is expansive, then $\mathcal{F}(U) = U$. If $\mathcal{F}$ is contractive, then $\mathcal{F}(\emptyset) = \emptyset$.*

Proposition 5. *Let $\mathcal{F}_1 : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{F}_2 : \mathcal{P}(U) \to \mathcal{P}(U)$ be expansive operators. Then the composition $\mathcal{F}_1 \odot \mathcal{F}_2$ is an expansive operator.*

Proposition 6. *Let $\mathcal{F}_1 : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{F}_2 : \mathcal{P}(U) \to \mathcal{P}(U)$ be contractive operators. Then the composition $\mathcal{F}_1 \odot \mathcal{F}_2$ is a contractive operator.*

2.3 Union Preserving Operators

If an operator operates on a union of two sets, and produces a result that can be obtained by taking the union of the operation on each of the sets, such an operator is called a union preserving operator.

Definition 7. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$. $\mathcal{F}$ is called union preserving if and only if*

$$\mathcal{F}(A \cup B) = \mathcal{F}(A) \cup \mathcal{F}(B)$$

The union preserving operators have the diagram as shown in Fig.1. It is clear that the union preserving operator has the same structure as a morphism.

$$
\begin{tikzcd}
A \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{F}\cup"'] & \mathcal{F}(A) \arrow[d, "\cup"] \\
B \arrow[r, "\mathcal{F}"] & \mathcal{F}(B)
\end{tikzcd}
$$

Fig. 1. Diagram of union preserving operators

Example 3. *Given a universal set domain:* $U = \{a, b, c\}$*. Let* $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ *be a union preserving operator with the following mapping.*

Table 3. Illustrates a union preserving operator $\mathcal{F}$ defined on the power set of $\{a, b, c\}$

S	$\mathcal{F}(S)$
$\emptyset$	$\{a\}$
$\{a\}$	$\{a, b\}$
$\{b\}$	$\{a, c\}$
$\{c\}$	$\{a\}$
$\{a, b\}$	$\{a, b, c\}$
$\{a, c\}$	$\{a, b\}$
$\{b, c\}$	$\{a, c\}$
$\{a, b, c\}$	$\{a, b, c\}$

The union preserving property implies that the entire mapping $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ *can be specified in terms of the subsets of singleton elements and the empty set. In the other words, we can use the singletons and the empty set to determine the whole mapping of the universal set as shown in Table 4.*

Table 4. Illustrates the set operator of Table 3 using only the rows of singletons and the row of empty set to determine any row of the whole table shown in Table 3

S	$\mathcal{F}(S)$
$\emptyset$	$\{a\}$
$\{a\}$	$\{a, b\}$
$\{b\}$	$\{a, c\}$
$\{c\}$	$\{a\}$

Using the union preserving property, any row of the full Table 3 can be generated from the table portrayed by Table 4. For example:

$$\begin{aligned}\mathcal{F}(\{a,b\}) &= \mathcal{F}(\{a\} \cup \{b\})\\ &= \mathcal{F}(\{a\}) \cup \mathcal{F}(\{b\})\\ &= \{a,b\} \cup \{a,c\}\\ &= \{a,b,c\}\end{aligned}$$

This means that we never need to store the whole table and whenever we have to perform dilation or erosion we can do it using the rows of table of the empty set and the singletons.

Proposition 7. *Let $\mathcal{F}_1 : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{F}_2 : \mathcal{P}(U) \to \mathcal{P}(U)$ be union preserving operators. Then $\mathcal{F}_1 \odot \mathcal{F}_2$ is union preserving.*

Proposition 8. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$. If $\mathcal{F}(A \cup B) \supseteq \mathcal{F}(A) \cup \mathcal{F}(B)$, then $\mathcal{F}$ is an increasing operator. If $\mathcal{F}(A \cap B) \subseteq \mathcal{F}(A) \cap \mathcal{F}(B)$, then $\mathcal{F}$ is an increasing operator.*

Proposition 9. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator, then $\mathcal{F}$ is an increasing operator.*

2.4 Intersection Preserving Operator

If an operator is applied to any intersection of two sets, and it produces a result that can be obtained by taking the intersection of the operation on each of the sets, then the operator is called an intersection preserving operator.

Definition 8. (Intersection preserving operator) *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$. $\mathcal{G}$ is called an intersection preserving operator if and only if*

$$\mathcal{G}(A \cap B) = \mathcal{G}(A) \cap \mathcal{G}(B)$$

The intersection preserving operators have the diagram as shown in Figure 2. It is clear that the intersection preserving operator has the same structure as a morphism.

Example 4. *Given a universal set domain: $U = \{a,b,c\}$. Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an intersection preserving operator with the following mapping:*

$$\begin{tikzcd} A \arrow[r, "\mathcal{G}"] \arrow[d, "\mathcal{G}\cap"'] & \mathcal{G}(A) \arrow[d, "\cap"] \\ B \arrow[r, "\mathcal{G}"] & \mathcal{G}(B) \end{tikzcd}$$

Fig. 2. Diagram of intersection preserving operators

Table 5. Illustrates an intersection preserving operator $\mathcal{G}$ defined on the power set of $\{a, b, c\}$

S	$\mathcal{G}(S)$
$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$
$\{b\}$	$\{c\}$
$\{c\}$	$\emptyset$
$\{a, b\}$	$\{c\}$
$\{a, c\}$	$\{b\}$
$\{b, c\}$	$\{c\}$
$\{a, b, c\}$	$\{a, b, c\}$

Proposition 10. *Let $\mathcal{G}_1 : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{G}_2 : \mathcal{P}(U) \to \mathcal{P}(U)$ be an intersection preserving operators. Then $\mathcal{G}_1 \odot \mathcal{G}_2$ is an intersection preserving operator.*

Proposition 11. *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a intersection preserving operator, then, $\mathcal{G}$ is an increasing operator.*

2.5 Set Dilation Operators and Set Erosion Operators

Expansive union preserving set operators are called set dilation operators. Contractive intersection preserving set operators are called set erosion operators.

Definition 9. *An operator $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ is called a set dilation operator on U if and only if $\mathcal{D}$ is*

- *Expansive:* $A \subseteq \mathcal{D}(A)$
- *Union preserving:* $\mathcal{D}(A \cup B) = \mathcal{D}(A) \cup \mathcal{D}(B)$

Definition 10. *An operator $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$ is called a set erosion operator on U if and only if $\mathcal{E}$ is*

- *Contractive:* $\mathcal{E}(A) \subseteq A$
- *Intersection preserving:* $\mathcal{E}(A \cap B) = \mathcal{E}(A) \cap \mathcal{E}(B)$

In mathematical morphology, dilation with a structuring element containing the origin, constitutes an instance of a set dilation operator, it is expansive and union preserving; erosion with a structuring element containing the origin, constitutes an instance of a set erosion operator, it is contractive and intersection preserving.

2.6 Dual Operators

Definition 11. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$. An operator $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ is called the dual operator to $\mathcal{F}$ if and only if $\mathcal{G}(A) = \mathcal{F}(A^c)^c$.*

In mathematical morphology, dilation and erosion are dual. Similarly, we have the flowing dual operators.

Proposition 12. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be the dual operators: $\mathcal{G}(A) = \mathcal{F}(A^c)^c$. Then*

(a) If $\mathcal{F}$ is an expansive operator, then $\mathcal{G}$ is a contractive operator.
(b) If $\mathcal{F}$ is a contractive operator, then $\mathcal{G}$ is an expansive operator.
(c) If $\mathcal{F}$ is an increasing operator, then $\mathcal{G}$ is an increasing operator.
(d) If $\mathcal{F}$ is an union preserving operator, then $\mathcal{G}$ is an intersection preserving operator.
(e) If $\mathcal{F}$ is an intersection preserving operator, then $\mathcal{G}$ is an union preserving operator.
(f) If $\mathcal{F}$ is a set dilation operator, then $\mathcal{G}$ is a set erosion operator.
(g) If $\mathcal{F}$ is a set erosion operator, then $\mathcal{G}$ is a set dilation operator.
(h) Let $\mathcal{F}_2 = \mathcal{F} \odot \mathcal{F}$ and $\mathcal{G}_2 = \mathcal{G} \odot \mathcal{G}$. Then $\mathcal{F}_2$ and $\mathcal{G}_2$ are duals.

2.7 Closing and Opening Operators

Definition 12. *An operator $\mathcal{T} : \mathcal{P}(U) \to \mathcal{P}(U)$ is called a closing operator on U if and only if $\mathcal{T}$ is*

- *Expansive: $A \subseteq \mathcal{T}(A)$*
- *Increasing: $A \subseteq B$ implies $\mathcal{T}(A) \subseteq \mathcal{T}(B)$*
- *Idempotent: $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$*

[28, 29]
An operator $\mathcal{Q} : \mathcal{P}(U) \to \mathcal{P}(U)$ is called an opening operator on U if and only if $\mathcal{Q}$ is

- *Contractive: $\mathcal{Q}(A) \subseteq A$*
- *Increasing: $A \subseteq B$ implies $\mathcal{Q}(A) \subseteq \mathcal{Q}(B)$*
- *Idempotent: $\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$*

[19, 31]

Example 5. *Given a universal set domain $U = \{a, b, c\}$. Let $\mathcal{T} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a closing operators, and $\mathcal{Q} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an opening operator with the following mappings shown in Table 6.*

Table 6. Illustrate a closing operator $\mathcal{T}$ and an opening operator $\mathcal{Q}$ defined on the power set of $\{a, b, c\}$

S	$\mathcal{T}(S)$	S	$\mathcal{Q}(S)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a\}$	$\{a\}$	$\emptyset$
$\{b\}$	$\{a, b\}$	$\{b\}$	$\{b\}$
$\{c\}$	$\{a, c\}$	$\{c\}$	$\{c\}$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{c\}$
$\{b, c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{b, c\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$

Proposition 13. *Let $\mathcal{E}$ be set erosion operator and let $\mathcal{D}$ be its dual operator, then $\mathcal{D} \odot \mathcal{E}$ is an opening operator.*

Proposition 14. *Let $\mathcal{D}$ be set dilation operator and let $\mathcal{E}$ be its dual operator, then $\mathcal{E} \odot \mathcal{D}$ is a closing operator.*

Definition 13. *Let $\mathcal{Q} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an opening operator and $\mathcal{T} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a closing operator. Let $A \subseteq U$. Then,*

- *A is open with respect to $\mathcal{Q}$ if and only if $A = \mathcal{Q}(A)$*
- *A is closed with respect to $\mathcal{T}$ if and only if $A = \mathcal{T}(A)$[21]*

A set A is closed with respect to the operator producing the closing. For example, given two different closing operators $\mathcal{T}_1$, and $\mathcal{T}_2$, a set A may be closed with respect to $\mathcal{T}_1$, but may not be closed with respect to $\mathcal{T}_2$. This behavior does not quite happen in the real analysis when we talk about a closed set because there is only one closing operator in real analysis, which has the concept of neighborhoods, and which defines the closed or open sets in a geometric way. Similarly, a set A may be open in accordance with the certain operator.

Proposition 15. *Let $\mathcal{T}$ be a closing operator on U. If B is a closed set with respect to $\mathcal{T}$ and $A \subseteq B$. Then $\mathcal{T}(A) \subseteq B$.*

Proposition 16. *Let $\mathcal{Q}$ be an opening operator on U. If B is an opened set with respect to $\mathcal{Q}$ and $A \subseteq B$. Then $\mathcal{Q}(A) \subseteq B$.*

The opening of a union is the union of the opening. The closing of an intersection is the intersection of the closing.

Proposition 17. *Let $\mathcal{Q}$ be an opening operator on U and let $\mathcal{T}$ be a closing operator on U. Let $A, B, C, D \subseteq U$ where A, B are open and C, D are closed. Then*

$$\mathcal{Q}(A \cup B) = \mathcal{Q}(A) \cup \mathcal{Q}(B)$$
$$\mathcal{T}(C \cap D) = \mathcal{T}(C) \cap \mathcal{T}(D)$$

Proposition 18. *Let $\mathcal{Q}$ be an opening operator on U and let $\mathcal{T}$ be a closing operator on U.Then $\mathcal{Q}(\emptyset) = \emptyset$ and $\mathcal{T}(U) = U$*

Proposition 19. *Let $\mathcal{Q}$ be an opening operator on U and let $\mathcal{T}$ be a closing operator on U. Let $A \subseteq U$. Define $\mathcal{C}(A) = \{X \mid \mathcal{T}(X) = X \text{ and } X \supseteq A\}$ and $\mathcal{O}(A) = \{X \mid \mathcal{Q}(X) = X \text{ and } X \subseteq A\}$ Then $\mathcal{C}(A) \neq \emptyset$ and $\mathcal{O}(A) \neq \emptyset$.*

The closing operator on a given set can be expressed as an intersection of sets related to the given set.

Proposition 20. *Let $\mathcal{T}$ be a closing operator on U. For any $A \subseteq U$ define $\mathcal{C}(A) = \{X \subseteq U \mid X = \mathcal{T}(X) \text{ and } X \supseteq A\}$. Then,*

$$\mathcal{T}(A) = \bigcap_{X \in \mathcal{C}(A)} X$$

The opening operator on a given set can be expressed as an union of sets related to the given set.

Proposition 21. *Let $\mathcal{Q}$ be an opening operator on U. For any $A \subseteq U$ define $\mathcal{O}(A) = \{X \subseteq U \mid X = \mathcal{Q}(X) \text{ and } X \subseteq A\}$. Then,*

$$\mathcal{Q}(A) = \bigcup_{X \in \mathcal{O}(A)} X$$

Proposition 22. *Let $\mathcal{Q}$ be an opening operator on U and let $\mathcal{T}$ be a closing operator on U. Let $A \subseteq U$. Define $\mathcal{O}(A) = \{X \subseteq U \mid X = \mathcal{Q}(X) \text{ and } X \subseteq A\}$ $\mathcal{C}(A) = \{X \subseteq U \mid X = \mathcal{T}(X) \text{ and } X \supseteq A\}$.*

If for every $Y \subseteq U$, $\mathcal{Q}(Y) = \mathcal{T}(Y^c)^c$, then $\mathcal{C}(A) = \{X \mid X^c \in \mathcal{O}(A^c)\}$ and $\mathcal{O}(A) = \{X \mid X^c \in \mathcal{C}(A^c)\}$.

Proposition 23. *Let $\mathcal{T} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{Q} : \mathcal{P}(U) \to \mathcal{P}(U)$ be dual operators: $\mathcal{Q}(A) = \mathcal{T}(A^c)^c$. Then $\mathcal{Q}$ is an opening operator if and only if $\mathcal{T}$ is a closing operator.*

Proposition 24. *Let $\mathcal{T} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{Q} : \mathcal{P}(U) \to \mathcal{P}(U)$ be dual opening and closing operators: $\mathcal{Q}(A) = \mathcal{T}(A^c)^c$. Then $A = \mathcal{T}(A)$ if and only if $\mathcal{Q}(A^c) = A^c$.*

The boundary of the set is a concept that is most often used in Euclidean and related spaces, spaces in which numbers play an essential role. Nevertheless, even in dealing with non-numeric sets, the concept of boundary can be defined by means of a closing operator.

Definition 14. *Let $\mathcal{T}$ be a closing operator on U and $A \subseteq U$. Then the boundary of A is defined by*

$$Bndry(A) = \mathcal{T}(A) \cap \mathcal{T}(A^c)$$

Proposition 25. *Let $\mathcal{T}$ be a closing operator on U and $A \subseteq U$. Then $\mathcal{T}(Bndry(A)) = Bndry(A)$.*

Proposition 26. *Let $\mathcal{T}$ be a closing operator on U and $A \subseteq U$. Then $\mathcal{T}(A) = A \cup Bndry(A)$.*

Proposition 27. *Let $\mathcal{T}$ be a closing operator on U and $A \subseteq U$. Then A is closed with respect to $\mathcal{T}$ if and only if $Bndry(A) \subseteq A$.*

Proposition 28. *Let $\mathcal{T} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ and $\mathcal{Q} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ be dual opening and closing operators: $\mathcal{Q}(A) = \mathcal{T}(A^c)^c$. Then $Bndry(A) = \mathcal{T}(A) \cup \mathcal{Q}(A)^c$*

Proposition 29. *Let $\mathcal{T} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ and $\mathcal{Q} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ be dual opening and closing operators: $\mathcal{Q}(A) = \mathcal{T}(A^c)^c$. Then*

$$\begin{aligned}
&\mathcal{T}(\mathcal{T}(A) \cap A^c) \subseteq Bndry(A) \\
&\mathcal{Q}(\mathcal{T}(A) \cap A^c) = \emptyset \\
&\mathcal{T}(A \cap \mathcal{Q}(A)^c) \subseteq Bndry(A) \\
&\mathcal{Q}(A \cap \mathcal{Q}(A)^c) = \emptyset
\end{aligned}$$

Proposition 30. *Let $\mathcal{T}$ be a closing operator on U and $\mathcal{Q}$ be its dual opening operator. Let $A \subseteq U$. Then A is open if and only if $Bndry(A) \subseteq A^c$.*

For any subset A, we can define its interior as the subset of all members of A that is separated from A^c. Likewise, we can define its interior boundary as the subset of all members of A that is connected to A^c. The dual concepts are the exterior of A and the exterior boundary of A. The exterior boundary of A is the subset of all members of A^c that connects to A. The exterior boundary of A is the subset of all members of A^c connected to A. These four concepts of interior, interior boundary, exterior boundary, and exterior allow the world to be partitioned into four pieces for each given set.

Part of an interior boundary may connect to the interior of A, and part may not. Thus, there is a part of interior boundary of A that is connected to A^c and connected to interior of A, and there is part of interior boundary of A that is connected to A^c but not connected to interior of A. The part that is connected to A^c but not connected to the interior of A is called a outlier of A.

A similar situation occurs with the exterior boundary of a set A. Part of exterior boundary of A may connect to the exterior of A, and part may not. Thus, there is a part of exterior boundary of A that is connected to A and connected to exterior boundary of A and there is a part of exterior boundary of A that is connected to A and not connected to exterior boundary of A . The part that is connected to A but not connected to exterior boundary of A is called a hole of A.

In fact, in mathematical morphology, dilation adds the exterior boundary to the set and erosion removes the interior boundary from the set. Dilation followed by erosion is called a closing. Erosion followed by dilation is called an opening. To find that a set has no holes and outliers if and only if dilation and erosion are inverses. Hence a set has no holes and no outliers if and only if $A \bullet S = (A \oplus S) \ominus S = A = (A \ominus S) \oplus S = A \circ S$ Similarly, for any set, we can define the closing and opening operators to remove the outliers and fill holes. For the general set operator, no holes and no outliers with respect to a closing operator $\mathcal{T}$ and its dual opening operator $\mathcal{Q}$ satisfies $\mathcal{T}(A) = A = \mathcal{Q}(A)$.

3 Inverse

In mathematical morphology, we know that dilation and erosion are dual, and can use its duality to construct closing and opening. However, applying the dual, we have to deal with the complement set, which may be very large. If we do it by the inverse, we may avoid large complement sets problems.

3.1 Inverse of Union Preserving Operator

For any union preserving operator, we can define its inverse, which is analogous to the pseudo inverse of matrix algebra.

Definition 15. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then we define its inverse by*

$$\mathcal{F}^{-1}(A) = \bigcup_{\{X \ \mid \ \mathcal{F}(X) \subseteq A\}} X$$

Example 6. *Based on Example 3, the union preserving operator $\mathcal{F}$ has the the mapping of $\mathcal{F}^{-1}$ as shown is Table 7:*

Table 7. Illustrate $\mathcal{F}^{-1}$ of the union preserving operator $\mathcal{F}$ defined on the power set of $\{a, b, c\}$ in Example 3

U	$\mathcal{F}^{-1}(U)$
$\{a\}$	$\emptyset \cup \{c\} = \{c\}$
$\{a, b\}$	$\emptyset \cup \{c\} \cup \{a\} \cup \{a, c\} = \{a, c\}$
$\{a, c\}$	$\emptyset \cup \{c\} \cup \{b\} \cup \{b, c\} = \{b, c\}$
$\{a, b, c\}$	$\emptyset \cup \{a\} \cup \{b\} \cup \{c\} \cup \{a, b\} \cup \{b, c\} \cup \{a, c\} \cup \{a, b, c\} = \{a, b, c\}$

Proposition 31. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then $\mathcal{F}^{-1}$ is an increasing operator.*

Proposition 32. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then $\mathcal{F}^{-1}(A \cup B) \supseteq \mathcal{F}^{-1}(A) \cup \mathcal{F}^{-1}(B)$*

Proposition 33. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then,*

$$\mathcal{F}(\mathcal{F}^{-1}(A)) \subseteq A$$
$$\mathcal{F}^{-1}(\mathcal{F}(A)) \supseteq A$$

Proposition 34. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then,*

$$\mathcal{F}^{-1}(\mathcal{F}(\mathcal{F}^{-1}(A))) = \mathcal{F}^{-1} \odot \mathcal{F} \odot \mathcal{F}^{-1}(A) = \mathcal{F}^{-1}(A)$$
$$\mathcal{F}(\mathcal{F}^{-1}(\mathcal{F}(A))) = \mathcal{F} \odot \mathcal{F}^{-1} \odot \mathcal{F}(A) = \mathcal{F}(A)$$

Proposition 35. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then $\mathcal{F}^{-1} \odot \mathcal{F}$ and $\mathcal{F} \odot \mathcal{F}^{-1}$ are idempotent. That is*

$$(\mathcal{F}^{-1} \odot \mathcal{F}) \odot (\mathcal{F}^{-1} \odot \mathcal{F}) = \mathcal{F}^{-1} \odot \mathcal{F}$$
$$(\mathcal{F} \odot \mathcal{F}^{-1}) \odot (\mathcal{F} \odot \mathcal{F}^{-1}) = \mathcal{F} \odot \mathcal{F}^{-1}$$

3.2 Inverse of Intersection Preserving Operator

Definition 16. *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an intersection preserving operator. Then we define its inverse $\mathcal{G}^{-1}$ by*

$$\mathcal{G}^{-1}(A) = \bigcap_{\{X \ | \ \mathcal{G}(X) \supseteq A\}} X$$

Example 7. *Based on Example 4, the intersection preserving operator $\mathcal{G}$ has the the mapping of $\mathcal{G}^{-1}$ as below in Table 8.*

Table 8. Illustrates the $\mathcal{G}^{-1}$ of the intersection preserving operator $\mathcal{G}$ defined in Table 5

U	$\mathcal{G}^{-1}(U)$
$\emptyset$	$\emptyset \cap \{a\} \cap \{b\} \cap \{c\} \cap \{a,b\} \cap \{a,c\} \cap \{b,c\} \cap \{a,b,c\} = \{a,b,c\} = \emptyset$
$\{b\}$	$\{a,c\} \cap \{a,b,c\} = \{a,c\}$
$\{c\}$	$\{b\} \cap \{a,b\} \cap \{b,c\} \cap \{a,b,c\} = \{b\}$
$\{a,b,c\}$	$\{a,b,c\}$

Proposition 36. *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an intersection preserving operator. Then $\mathcal{G}^{-1}$ is an increasing operator.*

Proposition 37. *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an intersection preserving operator. Then,*

$$(\mathcal{G} \odot \mathcal{G}^{-1})(A) \supseteq A$$
$$\mathcal{G}^{-1} \odot \mathcal{G}(A) \subseteq A$$

Proposition 38. *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an intersection preserving operator. Then,*

$$\mathcal{G}^{-1} \odot \mathcal{G} \odot \mathcal{G}^{-1} = \mathcal{G}^{-1}$$
$$\mathcal{G} \odot \mathcal{G}^{-1} \odot \mathcal{G} = \mathcal{G}$$

Proposition 39. *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$. If $\mathcal{G}$ is an intersection preserving operator, then $\mathcal{G} \odot \mathcal{G}^{-1}$ and $\mathcal{G}^{-1} \odot \mathcal{G}$ are idempotent.*

Proposition 40. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then $\mathcal{F}^{-1}$ is an intersection preserving operator.*

Proposition 41. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a intersection preserving operator. Then $\mathcal{F}^{-1}$ is an union preserving operator.*

3.3 Inverse of Set Dilation and Set Erosion Operator

Proposition 42. *Let $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a set dilation operator.Then $\mathcal{D}^{-1}$ is a set erosion operator.*

Proposition 43. *Let $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a set erosion operator.Then $\mathcal{E}^{-1}$ is a set dilation operator.*

Proposition 44. *Let $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a set dilation operator. Let $A \subseteq U$ and $B = \mathcal{D}^{-1} \odot \mathcal{D}(A)$. Then $C \supseteq B$ and $\mathcal{D}(C) = \mathcal{D}(A)$ imply $C = B$.*

Proposition 45. *Let $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a set erosion operator. Let $A \subseteq U$ and $B = \mathcal{E}^{-1} \odot \mathcal{E}(A)$. Then $C \subseteq B$ and $\mathcal{E}(C) = \mathcal{E}(A)$ imply $C = B$.*

4 Theorems of closing and opening operators

In mathematical morphology, dilation and erosion can define the closing and opening. Similarly, for any set, the union preserving operator and the intersection operator can construct the closing and opening.

Theorem 1. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a union preserving operator. Then $\mathcal{F}^{-1} \odot \mathcal{F}$ is a closing operator.*

Proof. Expansive: Let $A \subseteq U$. By proposition 33, $\mathcal{F}^{-1} \odot \mathcal{F}(A) \supseteq A$, so $\mathcal{F}^{-1} \odot \mathcal{F}(A)$ is expansive.

Increasing: By the proposition 9, $\mathcal{F}$ is union preserving, then it is increasing. By the Proposition 31, $\mathcal{F}$ is union preserving implies that $\mathcal{F}^{-1}$ is increasing. And by the proposition 2, the composition of increasing operators is increasing. Hence, $\mathcal{F}^{-1} \odot \mathcal{F}$ is increasing.

Idempotent: Finally, by proposition 35, $\mathcal{F}^{-1} \odot \mathcal{F}$ is idempotent. Now by the definition of the closing operator, $\mathcal{F}^{-1} \odot \mathcal{F}(A)$ is a closing operator.

Theorem 2. *Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be an intersection operator. Then $\mathcal{G}^{-1} \odot \mathcal{G}$ is an opening operator.*

Proof. Contractive: Let $A \subseteq U$. By proposition 37, $\mathcal{G}^{-1} \odot \mathcal{G}(A) \subseteq A$, so $\mathcal{G}^{-1} \odot \mathcal{G}(A)$ is contractive.

Increasing: By the proposition 11, $\mathcal{G}$ is intersection preserving implies it is increasing. By corollary 36, $\mathcal{G}$ is intersection preserving implies $\mathcal{G}^{-1}$ is increasing. And by the proposition 2, the composition of increasing operators is increasing. Hence, $\mathcal{G}^{-1} \odot \mathcal{G}(A)$ is increasing.

Idempotent: By proposition 39, $\mathcal{G}^{-1} \odot \mathcal{G}(A)$ is idempotent. Now by the definition of an opening operator, $\mathcal{G}^{-1} \odot \mathcal{G}(A)$ is an opening.

Example 8. Instance Illustrating Theorem 1 and 2 *Applying Theorem 1 with union preserving operator $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ in the Examples 3, 6, and applying the Theorem 2 with intersection preserving operator $\mathcal{G}$ and its inverse $\mathcal{G}^{-1}$ in the Examples 4, 7, we can define $\mathcal{F}^{-1} \odot \mathcal{F}$ and $\mathcal{G}^{-1} \odot \mathcal{G}$ as shown in Table9.*

Table 9. Illustrates $\mathcal{F}^{-1} \odot \mathcal{F}$ is a closing operator and $\mathcal{G}^{-1} \odot \mathcal{G}$ is an opening operator

A	$\mathcal{F}(A)$	$\mathcal{F}^{-1}(\mathcal{F}(A))$	A	$\mathcal{G}(A)$	$\mathcal{G}^{-1}(\mathcal{G}(A))$
$\emptyset$	$\{a\}$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a\}$	$\emptyset$	$\emptyset$
$\{b\}$	$\{a,c\}$	$\{b,c\}$	$\{b\}$	$\{c\}$	$\{b\}$
$\{c\}$	$\{a\}$	$\{c\}$	$\{c\}$	$\emptyset$	$\emptyset$
$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b\}$	$\{c\}$	$\{b\}$
$\{a,c\}$	$\{a,b\}$	$\{a,c\}$	$\{a,c\}$	$\{b\}$	$\{a,c\}$
$\{b,c\}$	$\{a,c\}$	$\{b,c\}$	$\{b,c\}$	$\{c\}$	$\{b\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

It is clear that $\mathcal{F}^{-1} \odot \mathcal{F}$ *is expansive, increasing, and idempotent, making it a closing operator. Similarly,* $\mathcal{G}^{-1} \odot \mathcal{G}$ *is contractive, increasing, and idempotent, making it a opening operator.*

Theorem 1 shows that we can construct an associated closing operator by only being given a union preserving operator. Theorem 2 shows that we can construct an opening operator by only be given an intersection preserving operator. In mathematical morphology, the morphological dilation can be rewritten as a union with translated structuring elements. The morphological erosion can be rewritten as the intersection with the structure elements. A union preserving operator or an intersection preserving operator always comes with some other features, may or may not are meaningful, but it can be considered to be its structure elements. Therefore, we can say the union preserving operator as the morphological dilation and the intersection operator as the morphological erosion, and they can construct the closing and opening.

In mathematical morphology, if set A is dilated by a structuring element B that does not contain the origin, then the dilated set may not have a single point in common with A. Similarly, if a set is eroded by a structuring element that does not contain the origin can lead to a result that has nothing in common with the set being eroded. In mathematical morphology, dilation with structuring elements containing the origin, constitute instances of a set dilation operator, expansive and union preserving; erosion with structuring elements containing the origin, constitute instances of a set erosion, contractive and intersection preserving. Adding the expansive and contractive properties to the Theorem1 and Theorem 2, we have the new theorems:

Theorem 3. *Let* $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ *be a set dilation operator. Then* $\mathcal{D}^{-1} \odot \mathcal{D}$ *is a closing operator.*

Theorem 4. *Let* $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$ *be a set erosion operator. Then* $\mathcal{E}^{-1} \odot \mathcal{E}$ *is an opening operator.*

Example 9. Instance Illustrating Theorem 3 *Given a universal set* $U = \{a, b, c\}$*. Let* $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ *be a set dilation operator as shown in Table 10.*

Table 10. Illustrates a set dilation operator $\mathcal{D}$ defined on the power set of $\{a, b, c\}$

A	$\mathcal{D}(A)$
$\emptyset$	$\emptyset$
$\{a\}$	$\{a, b\}$
$\{b\}$	$\{b, c\}$
$\{c\}$	$\{a, c\}$
$\{a, b\}$	$\{a, b, c\}$
$\{a, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\{a, b, c\}$
$\{a, b, c\}$	$\{a, b, c\}$

Given that $\mathcal{D}$ is a set dilation operator, it is a union preserving operator. We can define its inverse $\mathcal{D}^{-1}$ as shown in Table 11.

Table 11. Illustrates the inverse $\mathcal{D}^{-1}$ of the set dilation operator $\mathcal{D}$ defined in Table 10 is a contractive operator

U	$\mathcal{D}^{-1}(U)$
$\emptyset$	$\emptyset$
$\{a, b\}$	$\emptyset \cup \{a\}$
$\{a, c\}$	$\emptyset \cup \{c\} = \{c\}$
$\{b, c\}$	$\emptyset \cup \{b\} = \{b\}$
$\{a, b, c\}$	$\emptyset \cup \{a\} \cup \{b\} \cup \{c\} \cup \{a, b\} \cup \{a, c\} \cup \{b, c\} \cup \{a, b, c\} = \{a, b, c\}$

Now, we define the mapping $\mathcal{D}^{-1} \odot \mathcal{D}(A)$ as shown in Table 12.

Table 12. Illustrates $\mathcal{D}^{-1} \odot \mathcal{D}$ is a closing operator

A	$\mathcal{D}(A)$	$\mathcal{D}^{-1}(\mathcal{D}(A))$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a, b\}$	$\{a\}$
$\{b\}$	$\{b, c\}$	$\{b\}$
$\{c\}$	$\{a, c\}$	$\{c\}$
$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$

It is clear that $\mathcal{D}^{-1} \odot \mathcal{D}$ *is expansive, increasing, and idempotent, making it a closing operator.*

Example 10. Instance Illustrating Theorem 4 *Given a universal set* $U = \{a, b, c\}$. *Let* $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$ *be a set erosion operator as shown in Table 13.*

Table 13. Illustrates a set erosion operator $\mathcal{E}$ defined on the power set of $\{a, b, c\}$

A	$\mathcal{E}(A)$
$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$
$\{b\}$	$\emptyset$
$\{c\}$	$\emptyset$
$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{c\}$
$\{b, c\}$	$\emptyset$
$\{a, b, c\}$	$\{a, b, c\}$

Given $\mathcal{E}$ *is an intersection preserving operator, we can define its inverse* $\mathcal{E}^{-1}$ *as shown in Table 14.*

Table 14. Illustrates the $\mathcal{E}^{-1}$ of the set erosion operator $\mathcal{E}$ defined in Table 13

U	$\mathcal{E}^{-1}(U)$
$\emptyset$	$\emptyset \cap \{a\} \cap \{b\} \cap \{c\} \cap \{a, b\} \cap \{a, c\} \cap \{b, c\} \cap \{a, b, c\} = \{a, b, c\} = \emptyset$
$\{b\}$	$\{a, b\} \cap \{a, b, c\} = \{a, b\}$
$\{c\}$	$\{a, c\} \cap \{a, b, c\} = \{a, c\}$
$\{a, b, c\}$	$\{a, b, c\}$

Now, we define the mapping $\mathcal{E}^{-1} \odot \mathcal{E}(A)$ *as shown in Table 15.*

Table 15. Illustrates an opening operator defined by $\mathcal{E}^{-1} \odot \mathcal{E}(A)$

A	$\mathcal{E}(A)$	$\mathcal{E}^{-1} \odot \mathcal{E}(A)$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\emptyset$
$\{b\}$	$\emptyset$	$\emptyset$
$\{c\}$	$\emptyset$	$\emptyset$
$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,c\}$	$\{c\}$	$\{a,c\}$
$\{b,c\}$	$\emptyset$	$\emptyset$
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

It is clear that $\mathcal{E}^{-1} \odot \mathcal{E}(A)$ is contractive, increasing, and idempotent, making, it an opening operator.

We can see that for each set dilation operator, we can find a corresponding operator that is a set erosion operator, and visa-versa. If we are given a set dilation operator, we can find its dual, which is a set erosion operator, then by the Theorem 4, we can define an opening operator. In the other case, if we are given a set erosion operator, we can find its dual, which is a set dilation operator, then by the Theorem 3, we can define a closing operator. Base on these ideas, we have the following theorems:

Theorem 5. *Let $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$. If $\mathcal{D}$ is a set dilation operator, and its dual $\mathcal{E}(A) = \mathcal{D}(A^c)^c$ is a set erosion operator. Then $\mathcal{E}^{-1} \odot \mathcal{E}$ is an opening operator.*

Proof. *By Proposition (f) and Theorem 4.*

Theorem 6. *Let $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$. If $\mathcal{E}$ is a set erosion operator, and its dual $\mathcal{D}(A) = \mathcal{E}(A^c)^c$ is a set dilation operator. Then $\mathcal{D}^{-1} \odot \mathcal{D}$ is a closing operator.*

Proof. *By Proposition (g) and Theorem 3.*

Example 11. Instance Illustrating Theorem 5
Let $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ be the set dilation operator which we defined in Table 10 of the example of Theorem 3. If we take the complement of set A, and apply the operator $\mathcal{D}$ on the set A^c, then take the complement of $\mathcal{D}(A^c)$, we find that $(\mathcal{D}(A^c))^c$ is a set erosion operator as shown on Table 16.

Table 16. Illustrates a set erosion operator defined by $(\mathcal{D}(A^c))^c$

A	A^c	$\mathcal{D}(A^c)$	$(\mathcal{D}(A^c))^c$
$\emptyset$	$\{a,b,c\}$	$\{a,b,c\}$	$\emptyset$
$\{a\}$	$\{b,c\}$	$\{a,b,c\}$	$\emptyset$
$\{b\}$	$\{a,c\}$	$\{a,b,c\}$	$\emptyset$
$\{c\}$	$\{a,b\}$	$\{a,b,c\}$	$\emptyset$
$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b\}$
$\{a,c\}$	$\{b\}$	$\{b,c\}$	$\{a\}$
$\{b,c\}$	$\{a\}$	$\{a,b\}$	$\{c\}$
$\{a,b,c\}$	$\emptyset$	$\emptyset$	$\{a,b,c\}$

Given that $\mathcal{E}$ is a set erosion operator, it is an intersection preserving operator. We can define its inverse $\mathcal{E}^{-1}$ as shown in Table 17.

Table 17. Illustrates the $\mathcal{E}^{-1}$ of $\mathcal{E}(A) = (\mathcal{D}(A^c))^c$, which is a expansive operator

U	$\mathcal{E}^{-1}(U)$
$\emptyset$	$\emptyset \cap \{a\} \cap \{b\} \cap \{c\} \cap \{a,b\} \cap \{a,c\} \cap \{b,c\} \cap \{a,b,c\} = \emptyset$
$\{a\}$	$\{a,c\} \cap \{a,b,c\} = \{a,c\}$
$\{b\}$	$\{a,b\} \cap \{a,b,c\} = \{a,b\}$
$\{c\}$	$\{b,c\} \cap \{a,b,c\} = \{b,c\}$
$\{a,b,c\}$	$\{a,b,c\}$

Now, we define the mapping $\mathcal{E}^{-1} \odot \mathcal{E}(A)$ as shown in Table 18.

Table 18. Illustrates an opening operator defined by $\mathcal{E}^{-1} \odot \mathcal{E}(A)$

A	$\mathcal{E}(A)$	$\mathcal{E}^{-1} \odot \mathcal{E}(A)$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\emptyset$
$\{b\}$	$\emptyset$	$\emptyset$
$\{c\}$	$\emptyset$	$\emptyset$
$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,c\}$	$\{a\}$	$\{a,c\}$
$\{b,c\}$	$\{c\}$	$\{b,c\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

It is clear that $\mathcal{E}^{-1} \odot \mathcal{E}(A)$ is contractive, increasing, and idempotent, making it an opening operator.

Example 12. Instance Illustrating Theorem 6
Let $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a set erosion operator which we defined in Table 13 of the example of Theorem 4. If we will take the complement of set A, and apply the operator $\mathcal{E}$ on the set A^c, then take the complement of $\mathcal{E}(A^c)$, we find that $(\mathcal{E}(A^c))^c$ is set dilation operator:

Table 19. Illustrate an set dilation operator defined by $(\mathcal{E}(A^c))^c$

A	A^c	$\mathcal{E}(A^c)$	$(\mathcal{E}(A^c))^c$
$\emptyset$	$\{a,b,c\}$	$\{a,b,c\}$	$\emptyset$
$\{a\}$	$\{b,c\}$	$\emptyset$	$\{a,b,c\}$
$\{b\}$	$\{a,c\}$	$\{c\}$	$\{a,b\}$
$\{c\}$	$\{a,b\}$	$\{b\}$	$\{a,c\}$
$\{a,b\}$	$\{c\}$	$\emptyset$	$\{a,b,c\}$
$\{a,c\}$	$\{b\}$	$\emptyset$	$\{a,b,c\}$
$\{b,c\}$	$\{a\}$	$\emptyset$	$\{a,b,c\}$
$\{a,b,c\}$	$\emptyset$	$\emptyset$	$\{a,b,c\}$

Now, we define $\mathcal{D}(A) = (\mathcal{E}(A^c))^c$. It is easy to verify that $\mathcal{D}$ is a union preserving operator. Given $\mathcal{D}$ is a union preserving operator, we can define its inverse $\mathcal{D}^{-1}$ as shown in Table 20.

Table 20. Illustrates the $\mathcal{D}^{-1}$ of $\mathcal{D}(A) = (\mathcal{E}(A^c))^c$, which is a contractive operator

U	$\mathcal{D}^{-1}(U)$
$\emptyset$	$\emptyset$
$\{a,b\}$	$\emptyset \cup \{b\} = \{b\}$
$\{a,c\}$	$\emptyset \cup \{c\} = \{c\}$
$\{a,b,c\}$	$\emptyset \cup \{a\} \cup \{b\} \cup \{c\} \cup \{a,b\} \cup \{a,c\} \cup \{b,c\} \cup \{a,b,c\} = \{a,b,c\}$

Now, we define the mapping $\mathcal{D}^{-1} \odot \mathcal{D}(A)$:

Table 21. Illustrates a closing operator defined by $\mathcal{D}^{-1} \odot \mathcal{D}(A)$

A	$\mathcal{D}(A)$	$\mathcal{D}^{-1} \odot \mathcal{D}(A)$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{b\}$	$\{a,b\}$	$\{b\}$
$\{c\}$	$\{a,c\}$	$\{c\}$
$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

It is clear that $\mathcal{D}^{-1} \odot \mathcal{D}(A)$ is expansive, increasing, and idempotent, making it a closing operator.

In Theorem 3, we show that given a set dilation operator $\mathcal{D}$, we can define a closing operator $\mathcal{T}(A) = \mathcal{D}^{-1} \odot \mathcal{D}(A)$. In proposition 23, we prove that the closing and opening operators are dual. Then we can define a corresponding opening operator $\mathcal{Q}(A) = \mathcal{T}(A^c)^c$.

Similarly, in Theorem 4, we show that given a set erosion operator $\mathcal{E}$, we can define an opening operator $\mathcal{Q}(A) = \mathcal{E}^{-1} \odot \mathcal{E}(A)$. In proposition 23, we prove that closing and opening operators are dual. Then we can define a corresponding closing operator $\mathcal{T}(A) = \mathcal{Q}(A^c)^c$.

Base on the above ideas, we have the following theorems:

Theorem 7. *Let $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$. If $\mathcal{D}$ is a set dilation operator, we can define the closing operator $\mathcal{T}(A) = \mathcal{D}^{-1} \odot \mathcal{D}(A)$, then the corresponding opening operator is $\mathcal{Q}(A) = \mathcal{T}(A^c)^c$.*

Proof. *By Theorem 3 and Proposition 23 .*

Theorem 8. *Let $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$. If $\mathcal{E}$ is a set erosion operator, we can define the opening operator $\mathcal{Q}(A) = \mathcal{E}^{-1} \odot \mathcal{E}(A)$, then we can find the corresponding closing operator $\mathcal{T}(A) = \mathcal{Q}(A^c)^c$.*

Proof. *By Theorem 4 and Proposition 23 .*

Example 13. Instance Illustrating Theorem 7 *In the Example 9, given a set dilation operator $\mathcal{D} : \mathcal{P}(U) \to \mathcal{P}(U)$ we define the corresponding closing operator $\mathcal{T}(A) = \mathcal{D}^{-1} \odot \mathcal{D}(A)$. Now we can define the opening operator by its dual property as following:*

Table 22. Illustrate an opening operator $\mathcal{Q}(A)$ defined by $(\mathcal{T}(A^c))^c$

A	A^c	$\mathcal{T}(A^c)$	$\mathcal{Q}(A) = (\mathcal{T}(A^c))^c$
$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\emptyset$
$\{a\}$	$\{b, c\}$	$\{a, b, c\}$	$\emptyset$
$\{b\}$	$\{a, c\}$	$\{a, b, c\}$	$\emptyset$
$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\emptyset$
$\{a, b\}$	$\{c\}$	$\{c\}$	$\{a, b\}$
$\{a, c\}$	$\{b\}$	$\{b\}$	$\{a, c\}$
$\{b, c\}$	$\{a\}$	$\{a\}$	$\{b, c\}$
$\{a, b, c\}$	$\emptyset$	$\emptyset$	$\{a, b, c\}$

We can see that this opening operator is exactly the same as the opening operator we show in Example 11, which applied using the theorem 5.

Example 14. Instance illustrating Theorem 8 *In example 10, given a set erosion operator $\mathcal{E} : \mathcal{P}(U) \to \mathcal{P}(U)$, we defined the corresponding opening operator $\mathcal{Q}(A) = \mathcal{E}^{-1} \odot \mathcal{E}(A)$. Now we can define the closing operator by is dual property as following:*

Table 23. Illustrate a closing operator $\mathcal{T}(A)$ defined by $(\mathcal{Q}(A^c))^c$

A	A^c	$\mathcal{Q}(A^c)$	$\mathcal{T}(A) = (\mathcal{Q}(A^c))^c$
$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\emptyset$
$\{a\}$	$\{b, c\}$	$\emptyset$	$\{a, b, c\}$
$\{b\}$	$\{a, c\}$	$\{a, c\}$	$\{b\}$
$\{c\}$	$\{a, b\}$	$\{a, b\}$	$\{c\}$
$\{a, b\}$	$\{c\}$	$\emptyset$	$\{a, b, c\}$
$\{a, c\}$	$\{b\}$	$\emptyset$	$\{a, b, c\}$
$\{b, c\}$	$\{a\}$	$\emptyset$	$\{a, b, c\}$
$\{a, b, c\}$	$\emptyset$	$\emptyset$	$\{a, b, c\}$

We can see that this closing operator is exactly the same as the closing operator we showed in the example 12, which applied the theorem 6.

Theorem 5 and Theorem 7 show that we have two different methods to define opening operators by being given a set dilation operator. Theorem 6 and Theorem 8 show that we have two different methods to define closing operators by being given a set erosion operator. The interesting things are these two opening operators we defined by the two Theorem 5 and Theorem 7 are exactly the same operator, the two closing operators we defined by the Theorem 6 and Theorem 8 are also the same operator.

Definition 17. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$. The operator $\mathcal{F}$ is called the left adjoint to the operator $\mathcal{G}$ and $\mathcal{G}$ is called the right adjoint of the operator $\mathcal{F}$ if and only if for every pair of subsets $A, B \subseteq U$,*

$$\mathcal{F}(A) \subseteq B \text{ if and only if } A \subseteq \mathcal{G}(B)$$

Proposition 46. *Let $\mathcal{F}$ be a set dilation operator and $\mathcal{G}$ its dual. If $\mathcal{F}$ is the left adjoint to $\mathcal{G}$, then $\mathcal{F}^{-1} = \mathcal{G}$.*

Proposition 47. *Let $\mathcal{F}$ be a set dilation operator and $\mathcal{G}$ its dual. If $\mathcal{F}$ is the left adjoint to $\mathcal{G}$, then*

$$\begin{aligned} \mathcal{F} \odot \mathcal{G} \odot \mathcal{F} &= \mathcal{F} \\ \mathcal{G} \odot \mathcal{F} \odot \mathcal{G} &= \mathcal{G} \\ (\mathcal{F} \odot \mathcal{G}) \odot (\mathcal{F} \odot \mathcal{G}) &= \mathcal{F} \odot \mathcal{G} \\ (\mathcal{G} \odot \mathcal{F}) \odot (\mathcal{G} \odot \mathcal{F}) &= \mathcal{G} \odot \mathcal{F} \end{aligned}$$

Proposition 48. *Let $\mathcal{F}$ be a set dilation operator and $\mathcal{G}$ its dual. If $\mathcal{F}$ is the left adjoint to $\mathcal{G}$, then $\mathcal{G}^{-1} = \mathcal{F}$.*

In mathematical morphology, dilation and erosion with the same structuring element are adjoint and inverse of each other. Similarly, if the set dilation operator with its dual are adjoints, then the duals are inverses. Then, we can have the following new theorems.

Theorem 9. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a set dilation operator. Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be its dual, a set erosion operator. If $\mathcal{F}$ is the left adjoint to $\mathcal{G}$, then $\mathcal{G} \odot \mathcal{F}$ is a closing operator.*

Theorem 10. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ be a set dilation operator. Let $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ be its dual, which is a set erosion operator. If $\mathcal{F}$ is the left adjoint to $\mathcal{G}$, then $\mathcal{F} \odot \mathcal{G}$ is an opening operator.*

Proposition 49. *Let $\mathcal{F}$ be a set dilation operator and $\mathcal{G}$ its dual. Suppose $\mathcal{F}$ is the left adjoint to $\mathcal{G}$. Let $A, B \subseteq U$. Then $\mathcal{F} \odot \mathcal{G}(A) = A$ and $\mathcal{F} \odot \mathcal{G}(B) = B$ implies $\mathcal{F} \odot \mathcal{G}(A \cup B) = A \cup B$.*

Proposition 50. *Let $\mathcal{F} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{G} : \mathcal{P}(U) \to \mathcal{P}(U)$ are duals with $\mathcal{F}$ being the set dilation operator. Suppose*

$$\mathcal{F}(A) \subseteq B \text{ if and only if } A \subseteq \mathcal{G}(B)$$

Then

$$\mathcal{F} \odot \mathcal{F}(A) \subseteq B \text{ if and only if } A \subseteq \mathcal{G} \odot \mathcal{G}(B)$$

Now we can prove that the composition of an opening operator with it dual closing operator is idempotent and likewise the composition of a closing operator with its dual opening operator is idempotent. Recall that set operators are associative and so parenthesization is not necessary.

Proposition 51. *Let $\mathcal{F}$ be a set dilation operator and $\mathcal{G}$ its dual. Suppose $\mathcal{F}$ is the left adjoint to $\mathcal{G}$. Let $\mathcal{T} = \mathcal{G} \odot \mathcal{F}$ be the closing operator associated with $\mathcal{F}$ and $\mathcal{G}$ and let $\mathcal{Q} = \mathcal{F} \odot \mathcal{G}$ be the opening operator associated with $\mathcal{F}$ and $\mathcal{G}$. Then*

$$\mathcal{T} \odot \mathcal{Q} \odot \mathcal{T} \odot \mathcal{Q} = \mathcal{T} \odot \mathcal{Q}$$
$$\mathcal{Q} \odot \mathcal{T} \odot \mathcal{Q} \odot \mathcal{T} = \mathcal{Q} \odot \mathcal{T}$$

Proposition 52. *Let $\mathcal{T} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\mathcal{Q} : \mathcal{P}(U) \to \mathcal{P}(U)$ be dual opening and closing operators: $\mathcal{Q}(A) = \mathcal{T}(A^c)^c$. Then*

1. $\mathcal{T}(\mathcal{T} \odot \mathcal{Q}(A) \cap \mathcal{Q} \odot \mathcal{T} \odot \mathcal{Q}(A)^c) = \mathcal{T} \odot \mathcal{Q}(A) \cap \mathcal{Q} \odot \mathcal{T} \odot \mathcal{Q}(A)^c$
2. $\mathcal{Q}(\mathcal{T} \odot \mathcal{Q}(A) \cap \mathcal{Q} \odot \mathcal{T} \odot \mathcal{Q}(A)^c) = \emptyset$
3. $\mathcal{T}(\mathcal{T} \odot \mathcal{Q} \odot \mathcal{T}(A) \cap \mathcal{Q} \odot \mathcal{T}(A)^c) = \mathcal{T} \odot \mathcal{Q} \odot \mathcal{T}(A) \cap \mathcal{Q} \odot \mathcal{T}(A)^c$
4. $\mathcal{Q}(\mathcal{T} \odot \mathcal{Q} \odot \mathcal{T}(A) \cap \mathcal{Q} \odot \mathcal{T}(A)^c) = \emptyset$

5 Conclusion

We have shown how the important properties in mathematical morphology hold in a general setting of symbolic or non-numeric sets. We develop many set operators, including increasing operators, decreasing operators, expansive operators, contractive operators, union preserving operators, intersection preserving operators, set dilation operators and set erosion operators, dual operators, and adjoint operators. We also developed the theory of set operators on non-numeric sets to remove outliers and fill in holes.

The composition of the set erosion operator with its dual, the set dilation operator, results in an operator that is contractive, increasing, and idempotent. It is an operator that has the property of producing open sets. With such an operator, we can take a set that has paper shred garbage nearby or touching the set and operate on it to remove the garbage. This happens for arbitrary sets just in the analogous way that the opening operator of real analysis produces that subset of the original set where every point in the subset is an interior point with respect to the structuring elements. Opening a set that is opened just produces the open set. Opening is idempotent. Simply, we say a set erosion operator followed by its dual set dilation operator is an opening operator, which removes outliers.

The composition of the set dilation operator with its dual, the set erosion operator results in an operator that is expansive, increasing, and idempotent.

It is an operator that has the property of producing closed sets. With such an operator, we can take a set that has even many small holes and operate on it to produce a set in which the holes are eliminated. This happens for arbitrary sets just in the analogous way that the closing operator of real analysis produces a set that includes the original set plus all its limit points. Closing a set that is closed just produces the closed set. Closing is idempotent. Simply, we say a set dilation operator followed by its dual set erosion operator is a closing operator, which fills the missing holes.

Additionally, the original set dilation operator, has for its dual the set erosion operator, so it is the case that the closing operator has for its dual the opening operator. The composition of a closing operator with an opening operator is idempotent. Similarly the composition of an opening operator with a closing operator is idempotent. With the above properties, there is enough structure in a set dilation operator and its dual, to produce operator compositions that can fill holes and gaps in the observed data and eliminate paper shred garbage, thereby changing the observed data set into one whose pattern is closer to the pattern in the underlying population from which the observed data set was sampled.

We intend to apply set operators to text processing, working with dictionaries, thesaurus, and nets such as wordnet, and work on document understanding using set operators.

References

1. Georges Matheron. Random sets and integral geometry. 1975
2. Jean Serra.Image analysis and mathematical morphology. Academic Press, Inc., 1983
3. Banon GJ, Barrera J. Decomposition of mappings between complete lattices by mathematical morphology, part i. general lattices. Signal Processing, 30(3):299-327,1993
4. Banon GJ, Barrera J. Set mapping decomposition by mathematical morphology. chapter to be included in a book entitled "Mathematical Morphology: Theory and Hardware," to be edited by RM Haralick. 1990.
5. Stanley R Sternberg. Grayscale morphology. Computer vision, graphics,and image processing, 35(3):333–355, 1986
6. Henk JAM Heijmans. Morphological image operators, volume 4. Academic Press Boston, 1994
7. Heijmans HJ, Ronse C. The algebraic basis of mathematical morphology I. Dilations and erosions. Computer Vision, Graphics, and Image Processing. 1990 Jun 1;50(3):245-95.
8. A Haas, G Matheron, and J Serra. Morphologie mathematique et granuloetries en place. InAnnales des mines, volume 11, pages 7–3, 1967.
9. Robert M Haralick and Linda G Shapiro. Computer and robot vision, volume 1. Addison-wesley Reading, 1992
10. Edward R Dougherty and Charles Robert Giardina.Image processing: continuous to discrete, volume 1. Prentice-Hall Englewood Cliffs, NJ, 1987

11. Charles R Giardina and Edward R Dougherty.Morphological methods inimage and signal processing. Prentice-Hall, Inc., 1988
12. Robert M Haralick, Stanley R Sternberg, and Xinhua Zhuang. Image analysis using mathematical morphology.IEEE transactions on pattern analysisand machine intelligence, (4):532–550, 1987
13. Petros A Maragos. A unified theory of translation-invariant systems with applications to morphological analysis and coding of images. PhD thesis, Georgia Institute of Technology, 1985
14. Petros Maragos and RW Schafer. Morphological skeleton representation and coding of binary images.IEEE Transactions on Acoustics, Speech, and Signal Processing, 34(5):1228–1244, 1986
15. Petros Maragos. Tutorial on advances in morphological image processing and analysis.Optical engineering, 26(7):267623, 1987
16. Petros Maragos. Pattern spectrum and multiscale shape representation.IEEE Transactions on pattern analysis and machine intelligence, 11(7):701–716,1989
17. Petros Maragos. A representation theory for morphological image and signal processing. IEEE Transactions on Pattern Analysis and Machine Intelligence, 11(6):586–599, 1989
18. Petros Maragos and Ronald Schafer. Morphological filters–part i: Their set-theoretic analysis and relations to linear shift-invariant filters.IEEE Transactions on Acoustics, Speech, and Signal Processing, 35(8):1153–1169, 1987
19. John L Pfaltz and Josef Šlapal. Transformations of discrete closure systems. Acta Mathematica Hungarica, 138(4):386–405, 2013
20. John L Pfaltz. A category of discrete partially ordered sets. In Mid-AtlanticAlgebra Conf., George Mason Univ., Fairfax VA, 2004
21. John L Pfaltz and Robert E Jamison. Closure systems and their structure.Information Sciences, 139(3-4):275–286, 2001
22. John L Pfaltz. Closure lattices.Discrete Mathematics, 154(1-3):217–236,1996
23. John L Pfaltz and James C French. Multiple inheritance and the closure of set operators in class hierarchies. In INSTITUTE FOR PARALLEL COMPUTATION, UNIV. OF VIRGINIA, 1992
24. John L Pfaltz and John E Karro. Transformations of antimatroid closure spaces. In In 25th Combinatoric, Computing, and Graph Theory Conf.,Boca, 1998
25. Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D Lawson, Michael Mislove, and Dana S Scott.Continuous lattices and domains, volume 93. Cambridge university press, 2003
26. Gabriele Castellini, Jürgen Koslowski, and George E Strecker. Closure operators and polarities. Annals of the New York Academy of Sciences, 704(1):38–52, 1993
27. G Castellini and D Hajek. Closure operators and connectedness. Topology and its Applications, 55(1):29–45, 1994
28. Birkhoff G. Lattice theory.Providence,Rhode Island: American Mathematical Society. 1967.
29. Everett CJ. Closure operators and Galois theory in lattices. Transactions of the American Mathematical Society. 1944 May 1;55(3):514-25.
30. Marcel Erné, Jürgen Koslowski, Austin Melton, and George E Strecker. Aprimer on galois connections. Annals of the New York Academy of Sciences,704(1):103–125, 1993
31. Jean Serra, Image Analysis and Mathematical Morphology, Volume 2: Theoretical Advances, New York: Academic Press,1988